# On the Complexity of Inferring Rooted Evolutionary Trees

Jesper Jansson [a]

[a] *Dept. of Computer Science, Lund University, Box 118, 221 00 Lund, Sweden,*
`Jesper.Jansson@cs.lth.se`

**Abstract**

We prove that the maximum inferred local consensus tree problem is NP-complete, thus resolving an open question from [3].

## 1 Introduction

An *evolutionary tree* is an unordered tree whose leaves are in one-to-one correspondence with a set of species. Internal nodes represent common ancestors, and the branching structure reflects the species' evolutionary relationships. In some settings, the data does not uniquely determine a root, which leads to unrooted (as opposed to rooted) trees.

Traditional methods for evolutionary tree construction can be divided into character-based, distance-based, and maximum likelihood methods [7,8,10]. Recently, methods for inferring evolutionary history from information concerning local topological relationships between the species have been proposed and analyzed. Among these are quartet methods [5,9] which first compute unrooted topologies for subsets of cardinality four of the species and then combine them to form an unrooted evolutionary tree, and rooted consensus methods [1,3,4,6].

Aho *et al.* [1] studied the problem of inferring a rooted tree from a set of constraints on lowest common ancestor relations. They showed how to decide whether an instance admits a solution that is consistent with all of the constraints, and if so, how to construct it, in $O(mn \log m)$ time, where $m$ is the number of constraints and $n$ the number of species. An even faster implementation restricted to constraints in the form of rooted, unordered, binary trees on three species was later given by Henzinger *et al.* [4]. However, data obtained experimentally often contains errors, implying that there usually will not exist a tree consistent with *all* of the constraints. A single erroneous constraint in the input might result in these algorithms returning the null tree. Therefore, [3] introduced optimization versions of the problem, called MICT and MILCT. [3] proved that MICT is NP-complete and proposed some approximation algorithms for MICT and MILCT, but left the computational complexity of MILCT as an open problem. In this paper, we prove that MILCT is NP-complete, obtaining a shorter NP-completeness proof for MICT in the process.

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## 2 Problem Definitions and Results

Let $S$ be a set of elements. An *LCA constraint on* $S$ is a constraint of the form $\{i, j\} < \{k, l\}$, where $i, j, k, l \in S$, which specifies that the lowest common ancestor of $i$ and $j$ is a proper descendant of the lowest common ancestor of $k$ and $l$. An LCA constraint of the form $\{i, j\} < \{i, k\}$ is called a *3-leaf constraint on* $S$; it uniquely determines the relative topology of $i, j$, and $k$, and is written as $(\{i, j\}, k)$ for short. A rooted tree with leaves distinctly labeled by elements in $S$ and an LCA constraint on $S$ which is satisfied in the tree are *consistent* with each other.

The *maximum inferred consensus tree problem* is to construct a rooted tree that is consistent with as many LCA constraints as possible from a given set. The corresponding decision problem is:

**The maximum inferred consensus tree problem (MICT)**

**Instance:** Finite set $U$, set $V$ of LCA constraints on $U$, positive integer $L \leq |V|$.

**Question:** Is there a rooted tree that is consistent with $L$ of the constraints in $V$?

A special case of MICT occurs when each constraint is a 3-leaf constraint:

**The maximum inferred local consensus tree problem (MILCT)**

**Instance:** Finite set $S$, set $T$ of 3-leaf constraints on $S$, positive integer $K \leq |T|$.

**Question:** Is there a rooted tree that is consistent with $K$ of the constraints in $T$?

To determine the computational complexity of MILCT, it will be useful to know that the following problem is NP-complete (problem [MS2] in [2]):

**Cyclic Ordering**

**Instance:** Finite set $A$, collection $C$ of ordered triples $(a, b, c)$ of distinct elements from $A$.

**Question:** Is there a one-to-one function $f : A \rightarrow \{1, 2, ..., |A|\}$ such that, for each $(a, b, c) \in A$, we have either $f(a) < f(b) < f(c)$, or $f(b) < f(c) < f(a)$, or $f(c) < f(a) < f(b)$?

We are now ready to prove the main result.

**Theorem 1** *MILCT is NP-complete.*

**PROOF.** MILCT is in NP since verifying if there exists a rooted tree that is consistent with a given subset of $T$ can be done in polynomial time with the algorithm of Aho *et al.* [1].

To show the NP-hardness of MILCT, we give a polynomial-time reduction from Cyclic Ordering. Given an instance $(A, C)$ of Cyclic Ordering, let $S =$

$A \cup \{x_0, x_1, x_2, ..., x_{|C|}\}$ and $K = \frac{|A| \cdot (|A|-1)}{2} + 2 \cdot |C|$. For each $a, b \in A$ with $a \neq b$, include the two constraints $(\{x_0, a\}, b)$ and $(\{x_0, b\}, a)$ in $T$. Next, for every $i$ in $\{1, 2, ..., |C|\}$, add to $T$ the three constraints $(\{x_i, a\}, b)$, $(\{x_i, b\}, c)$, and $(\{x_i, c\}, a)$, where $(a, b, c)$ is the $i$th ordered triple in $C$. Note that at most one of $(\{x_0, a\}, b)$ and $(\{x_0, b\}, a)$ and at most two of $(\{x_i, a\}, b)$, $(\{x_i, b\}, c)$, and $(\{x_i, c\}, a)$ can be consistent with any rooted tree, so the number of constraints in $T$ that can be satisfied must be $\leq K$.

Claim: $(A, C)$ has a cyclic ordering if and only if there exists a rooted tree that is consistent with $K$ of the constraints in $T$.

Proof of claim: Suppose the answer to the Cyclic Ordering instance is yes. Then there exists a one-to-one function $f : A \to \{1, 2, ..., |A|\}$ such that, for each ordered triple $(a, b, c) \in C$, we have either $f(a) < f(b) < f(c)$, or $f(b) < f(c) < f(a)$, or $f(c) < f(a) < f(b)$. We can construct a rooted tree consistent with $K$ constraints as in Fig. 1.

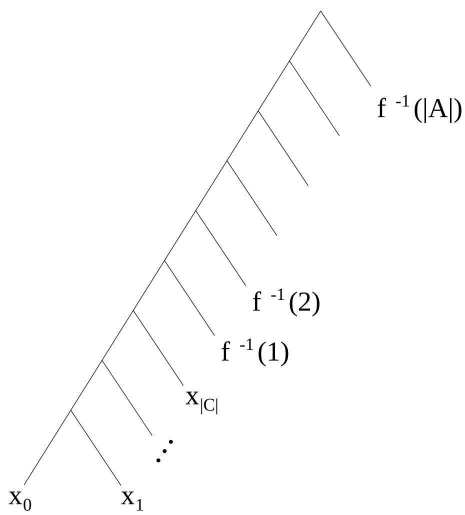


Fig. 1.

If $f(\alpha_i) < f(\beta_i) < f(\gamma_i)$ for the $i$th ordered triple in $C$, then $(\{x_i, \alpha_i\}, \beta_i)$ and $(\{x_i, \beta_i\}, \gamma_i)$ are consistent with the tree. Also, for each pair $a, b \in A$ with $a \neq b$, exactly one of $(\{x_0, a\}, b)$ and $(\{x_0, b\}, a)$ is consistent with the tree. Thus, the tree is consistent with $2 \cdot |C| + \frac{|A| \cdot (|A|-1)}{2}$ of the constraints in $T$.

Conversely, suppose there exists a rooted tree $R$ consistent with $\frac{|A| \cdot (|A|-1)}{2} + 2 \cdot |C|$ of the constraints. At most $\frac{|A| \cdot (|A|-1)}{2}$ constraints of type $(\{x_0, a\}, b)$ and at most $2 \cdot |C|$ constraints of type $(\{x_i, a\}, b)$ with $i \neq 0$ can be consistent with $R$, so $R$ must be consistent with precisely this many constraints of each type, respectively. $\frac{|A| \cdot (|A|-1)}{2}$ constraints of the former type can only be satisfied if the subtree of $R$ induced by $A \cup \{x_0\}$ is a rooted caterpillar whose root is the parent of a leaf and an internal node, and one of the two leaves at maximum distance from the root is labeled $x_0$ (otherwise, for some pair $a, b \in A$, neither

$(\{x_0, a\}, b)$ nor $(\{x_0, b\}, a)$ would be consistent with $R$). For each $a \in A$, let $f(a)$ be the number of internal nodes on the path from $a$ to $x_0$ in the subtree of $R$ induced by $A \cup \{x_0\}$. Next, because of the constraints of the second type, for every ordered triple $(a, b, c) \in C$, exactly two of the three corresponding constraints in $T$ are consistent with $R$ (if, for some ordered triple, just one constraint was consistent with $R$, then the number of constraints of this type consistent with $R$ could not add up to $2 \cdot |C|$). Therefore, either (1) $a$ is closer to $x_0$ than $b$ is to $x_0$ and $b$ is closer to $x_0$ than $c$ is to $x_0$, implying $f(a) < f(b) < f(c)$, or (2) $b$ is closer to $x_0$ than $c$ is to $x_0$ and $c$ is closer to $x_0$ than $a$ is to $x_0$, implying $f(b) < f(c) < f(a)$, or (3) $c$ is closer to $x_0$ than $a$ is to $x_0$ and $a$ is closer to $x_0$ than $b$ is to $x_0$, implying $f(c) < f(a) < f(b)$. □

**Corollary 2** *MICT is NP-complete.*

**PROOF.** MICT is in NP because the algorithm of Aho *et al.* [1] can check any given subset of the LCA constraints for consistency in polynomial time. MICT is NP-hard since it admits a direct reduction from MILCT; just replace each 3-leaf constraint $(\{a, b\}, c)$ in the given instance by $\{a, b\} < \{a, c\}$. □

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